UNIPRIMITIVE GROUPS OF DEGREE $\frac{1}{2}n(n-1)$

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ABSTRACT

The goal of this paper is to study the order and sub-groups of a uniprimitive groups of degree $\frac{1}{2}n(n-1)$, where n is an integer and greater than 4 $(n \in Z > 4)$, where the sub-degrees, sub-orbit lengths of these groups are also investigated in this work. Along with carrying the survey on the area of the groups and their properties it is also pointed out that the particular generalizations will be useful for the further investigations and interpretation of uniprimitive groups. In this paper we prove the result which is based on the properties of uniprimitive groups.

Keywords: ***Symmetric Group, uniprimitive, primitive, sub-degrees, sub-orbits lengths, and permutation.***

INTRODUCTION

The study of primitive group having small degree has a wide and well known past. This short introduction is based on the historical background and some important developments on primitive groups by some of the authors are given in this presentation. The study of permutation groups of degree less than 1000 has been discussed in [5]. Later on the research on the Primitive Permutation Groups of Degree Less Than 4096 was explored and compared with past studies by Hanna J. Coutts et al [3]. P. J. Cameron [9] has given the idea of Finite Permutation groups and Finite Simple groups in wide aspect.

The study of permutation group is essential for exploring advanced study on uniprimitive groups. Before we present our results (axioms, theorems) on uniprimitive groups, it is necessary to know about few famous related necessary results on Permutation groups.

Definition: A subgroup of the symmetric group G on a set S is called a permutation group or group of permutations.

Definition: For any non-empty set G, the set of all even permutations of G_n is a subgroup of G_n, hence the set of all even permutations of G_n is called the irregular group on n elements, and will be denoted by G_n.

We refer the reader for the overview of uniprimitive groups of degree 120 to [6] where such types of properties for these groups are investigated. Our goal is to generalize these results to the uniprimitive groups of degree $\frac{1}{2}n(n-1)$. For basic definitions, notations and preliminary results in permutation groups, we refer the reader to H. Wielandt [4] and for the representation theory and group characters the reader is referred to M. Burrow [8]. We propose the following main result:

THEOREM:

The symmetric group S_n is uniprimitive of degree $\frac{1}{2}n(n-1)$ acting on unordered sets, where $S_n + 1$ is not square. The sub-degrees are $1,\ (n-1),\ \frac{1}{2}n(n-3)$ and sub-orbit lengths are

$$1,\ 2(n-2),\ \frac{1}{2}(n-2)(n-3).$$

PROOF:

Consider $G = S_n$ acting naturally on $\Phi = \{1, 2, 3, \cdots, n\}$, then Facts on $\Omega = \left\{A \subseteq \Phi \middle|^{*} |A| = 2\right\}$, the set of $\left|\begin{matrix} n \\ 2 \end{matrix}\right. = \frac{1}{2}n(n-1)$ unordered sets i.e.

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$\Omega = \{1, 2\}, \{1, 3\}, \cdots, \{1, n\}, \{2, 3\}, \{2, 4\}, \cdots,$
$\{2, n\}, \cdots, \cdots, \{n-1, n\}$

G is transitive but not doubly transitive on Ω, because there is no element in G, this takes $\{1, 2\}$ to $\{1, 2\}$ and $\{1, 3\}$ to $\{4\ 5\}$. We find orbits of $G\ \{1, 2\}$, i.e. sub-orbits of G on Ω. Let $g \in G$, then $g \in G_{\{1, 2\}}$, iff $\{1,2\}\hat{g} = \{1,2\}$, equivalently g either fixes or transpose 1 and 2, and may induce any permutation on $\{3, 4, \cdots, n\}$.

Hence,

$$\{1, 3\}, \{1, 2\} = \{1, 3\}, \{1, 4\}, \cdots,$$
$$\{1, n\}, \{2, 3\}, \{2, 4\}, \cdots, \{2, n\}$$

this is the set of all unordered sets containing exactly one of 1 and 2. $\{3, 4\}\, G_{\{1, 2\}} = \begin{Bmatrix} \{3, 4\}, \{3, 5\}, \cdots, \{3, n\}, \{4, 5\}, \\ \{4, 6\}, \cdots, \{4, n\}, \cdots, \{n-1, n\} \end{Bmatrix}$

Hence, the orbits of $G_{\{1, 2\}}$ or sub-orbit of G have lengths $1,\ 2(n-2),\ \frac{1}{2}(n-2)(n-3)$. Now we prove that G is primitive. If G is not primitive (imprimitive), then G has a block of non-primitives ψ. The length of ψ divides $|\Omega| = \frac{1}{2}n(n-1)$ and is a union of some orbits of $G_{\{1, 2\}}$, not true orbits of $G_{\{1, 2\}}$ which have the length $1,\ 2(n-2),\ \frac{1}{2}(n-2), \frac{1}{2}(n-2)(n-3)$. Therefore G is primitive and hence unimprimitive.

Again we find the degrees of irreducible characters G, i.e. sub-degrees of G.

Let $\{{}^f\alpha\}, \Delta(\alpha), \Gamma(\alpha)$, be sub-orbits of G with lengths $1,\ 2(n-2),\ \frac{1}{2}(n-2)(n-3)$ respectively $(\alpha \in \Omega)$. Then we recall the following result from the lemma 2 of Higman [4],

$$|\Delta(\alpha)\,\eta(\beta)| = \begin{cases} \lambda & for \quad \beta \in \Delta(\alpha) \\ \mu & for \quad \beta \in \Gamma(\alpha) \end{cases}$$

By lemma 5 of D. G. Higman [2],

$$\mu\ell = k(k-\lambda-1)$$

putting $k = |\Delta(\alpha)| = 2(n-2)$ and

$$\ell = |\Gamma(\alpha)| = \frac{1}{2}(n-2)(n-3)$$

then we get

$$\frac{1}{2}(n-2)(n-3)\mu = 2\,(n-2)\{2\,(n-2) - \lambda - 1\}$$
$$\Rightarrow (n-3)\,\mu + 4\lambda = 4(2n-5)$$

The possible solutions are

Case(i):

$\mu = 4, \quad \lambda = n-2$,

Now $|G|$ is even because $|\Omega| = 120$ divided $|G|$.

Hence by II lemma 7 of [2],

$d = (\lambda-\mu)^2 + 4\,(k-\mu)$ is perfect square.

Case(ii)

if $\mu = 8, \lambda = 1$, and $k = 2(n-2)$ then by simplification , we get

$$d = 49 + 8n - 48$$
$$\Rightarrow d = 8n + 1$$

Therefore d is not perfect square.

Therefore, the selections $\mu = 8,\ \lambda = 1$, and $k = 2(n-2)$, is not possible.

Case (iii)

Now If we take $\mu = 4, \quad \lambda = n-2$ and $k = 2(n-2)$,

Then

$$d = (n-6)^2 + 4(2n-8)$$
$$= (n-1)^2$$

Therefore d is perfect squared.

By (2, p. 150)

$$f_2, f_3 = \frac{2k + (\lambda - \mu)(k+\ell) \mp \sqrt{d\,(k+\ell)}}{\mp\sqrt{d}}$$

Putting the values of $d, k, \ell, \lambda, \mu,$ then we get

$$f_2 = (n-1) \text{ and } f_3 = \frac{1}{2}n(n-3)$$

Hence proof the statement of theorem.

CONCLUSION:

Study of uniprimitive groups is given in this paper where we have generalized the degree of uniprimitive groups from 120 to $\frac{1}{2}n(n-1)$, with $n \in Z > 4$, moreover we have investigated the sub-degrees, sub-orbit lengths of the groups discussed in this paper. The results and the possible case study of the problem is given in the main theorem.

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